

## NOTE ON A CLASS OF INTEGRAL OPERATORS OF SZEGÖ TYPE

JONGHO YANG

ABSTRACT. We define new integral operators on the Hardy space similar to Szegő projection. We consider a relationship between these Szegő type operators and the partial sum of Taylor series on the Hardy space.

### 1. Introduction

Let  $\mathbf{C}^n$  denote the Euclidean space of complex dimension  $n$ . The inner product on  $\mathbf{C}^n$  is given by

$$\langle z, w \rangle := z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

where  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$ , and the associated norm is  $|z| := \sqrt{\langle z, z \rangle}$ . The unit ball in  $\mathbf{C}^n$  is the set

$$\mathbf{B}_n := \{z \in \mathbf{C}^n : |z| < 1\}$$

and its boundary is the unit sphere

$$\mathbf{S}_n := \{z \in \mathbf{C}^n : |z| = 1\}.$$

In case  $n = 1$ , we denote  $\mathbf{D}$  in place of  $\mathbf{B}_1$ .

Let  $\sigma_n$  be the normalized surface measure on  $\mathbf{S}_n$ .

For  $0 < p < \infty$ , the Hardy space  $H^p(\mathbf{B}_n)$  is the space of all holomorphic function  $f$  on  $\mathbf{B}_n$  for which the “norm”

$$\|f\|_{H^p} := \left\{ \sup_{0 < r < 1} \int_{\mathbf{S}_n} |f(r\zeta)|^p d\sigma_n(\zeta) \right\}^{1/p}$$

is finite. As is well-known, the space  $H^p(\mathbf{B}_n)$  equipped with the norm above is a Banach space for  $1 \leq p < \infty$ . On the other hand, it is a

---

Received April 07, 2020; Accepted October 23, 2020.

2010 Mathematics Subject Classification: Primary 30H10; Secondary 47B38, 46B70.

Key words and phrases: Hardy space, Szegő projection, Coefficient multiplier.

complete metric space for  $0 < p < 1$  with respect to the translation-invariant metric  $(f, g) \mapsto \|f - g\|_{H^p}^p$ .

For a function  $f$  in  $H^p(\mathbf{B}_n)$ , it is known that  $f$  have a radial limit function  $f^*$  almost everywhere on  $\mathbf{S}_n$ . Here, the radial limit function  $f^*$  of  $f$  is defined by

$$f^*(\zeta) := \lim_{r \rightarrow 1^-} f(r\zeta)$$

provided that the limit exists for  $\zeta \in \mathbf{S}_n$ . Moreover the mapping  $f \mapsto f^*$  is an isometry from  $H^p(\mathbf{B}_n)$  into  $L^p(\mathbf{S}_n, d\sigma_n)$ . Consequently, each  $H^p(\mathbf{B}_n)$  can be identified with a closed subspace of  $L^p(\mathbf{S}_n, d\sigma_n)$ .

Since  $H^2(\mathbf{B}_n)$  can be identified with a closed subspace of  $L^2(\mathbf{S}_n, d\sigma_n)$ , there exists an orthogonal projection from  $L^2(\mathbf{S}_n, d\sigma_n)$  onto  $H^2(\mathbf{B}_n)$ . By using a reproducing kernel function, which is called the Szegő kernel, we also obtain a function  $f$  from its radial limit function  $f^*$ . More precisely,

$$(1.1) \quad f(z) = T[f](z) := \int_{\mathbf{S}_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma_n(\zeta)$$

for  $f \in H^2(\mathbf{B}_n)$ . We usually call this integral operator as the Szegő projection. It is well known that for  $1 < p < \infty$  the Szegő projection maps  $L^p(\mathbf{S}_n, d\sigma_n)$  boundedly onto  $H^p(\mathbf{B}_n)$ . For more details, we refer the classical text books [1, 2, 4].

For a holomorphic function  $f$  on  $\mathbf{B}_n$  with Taylor series

$$f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha},$$

we define  $N$ -th partial sum of  $f$  by

$$(1.2) \quad S_N f(z) := \sum_{|\alpha| \leq N} c_{\alpha} z^{\alpha}$$

for a positive integer  $N$ . In [3], it is known that  $S_N f$  converges to  $f$  in  $H^p(\mathbf{B}_n)$  for  $1 < p < \infty$ .

In this paper we consider a class of integral operators defined by

$$(1.3) \quad T_{m,N}[f](z) := \int_{\mathbf{S}_n} \frac{\langle z, \zeta \rangle^{m+N}}{(1 - \langle z, \zeta \rangle)^m} f^*(\zeta) d\sigma_n(\zeta)$$

for  $m = 1, 2, \dots, n$  and a positive integer  $N$ . With this operators we give a relationship between  $S_N$  and  $T_{m,N}$  on  $H^p(\mathbf{B}_n)$ . More precisely we give the following theorem.

**THEOREM 1.1.** *Let  $1 < p < \infty$  and  $N$  be a positive integer. For  $f$  in  $H^p(\mathbf{B}_n)$ ,*

$$f - S_N f = \sum_{m=1}^n c_{m,n,N} T_{m,N}[f],$$

where  $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$ .

We note that the Szegő projection  $T$  defined in (1.1) is bounded only if  $p > 1$ . It is known that  $T$  is an unbounded operator on  $L^1(\mathbf{S}_n, d\sigma_n)$ . For  $m = n$ ,  $T_{m,N}$  has a similar growth condition with  $T$ . Thus the range of  $p$  in Theorem 1.1 is restricted.

### 2. Preliminary results

We use the conventional multi-index notation. For a multi-index

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

with nonnegative integers  $\alpha_i$ , the following are common notations;

$$|\alpha| := \alpha_1 + \dots + \alpha_n,$$

$$\alpha! := \alpha_1! \cdots \alpha_n!$$

For  $z \in \mathbf{C}^n$ , the monomial is defined as

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

At first, we show that the Szegő type operators  $T_{m,N}$  defined in (1.3) are actually coefficient multipliers.

**PROPOSITION 2.1.** *Let  $m, N$  be positive integers with  $1 \leq m \leq n$ . For a multi-index  $\alpha$ , there exists  $\lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|)$  such that*

$$T_{m,N}[\zeta^\alpha](z) = \lambda_\alpha z^\alpha.$$

*Proof.* From the definition of  $T_{m,N}$ , we have

$$T_{m,N}[\zeta^\alpha](z) = \int_{\mathbf{S}_n} \frac{\langle z, \zeta \rangle^{m+N} \zeta^\alpha}{(1 - \langle z, \zeta \rangle)^m} d\sigma_n(\zeta)$$

for a multi-index  $\alpha$ . Note that

$$\frac{1}{(1 - \langle z, \zeta \rangle)^m} = \sum_{k=0}^{\infty} \binom{k + m - 1}{k} \langle z, \zeta \rangle^k.$$

Since the monomials are orthogonal on  $L^2(\mathbf{S}_n, d\sigma_n)$ ; see [2, Proposition 1.4.8], we have  $T_{m,N}[\zeta^\alpha](z) = 0$  if  $|\alpha| < m + N$ . In case of  $|\alpha| \geq m + N$ , we have

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \int_{\mathbf{S}_n} \sum_{k=0}^{\infty} \binom{k+m-1}{k} \langle z, \zeta \rangle^{k+m+N} \zeta^\alpha d\sigma_n(\zeta) \\ &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \int_{\mathbf{S}_n} \langle z, \zeta \rangle^{|\alpha|} \zeta^\alpha d\sigma_n(\zeta). \end{aligned}$$

Expanding the term inside the above integral as

$$\langle z, \zeta \rangle^{|\alpha|} = \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} z^\beta \bar{\zeta}^\beta,$$

we obtain that

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{|\alpha|!}{\alpha!} z^\alpha \int_{\mathbf{S}_n} |\zeta^\alpha|^2 d\sigma_n(\zeta) \\ &= \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} z^\alpha, \end{aligned}$$

see [2, Proposition 1.4.9] for the last equality. Putting  $\lambda_\alpha$  as

$$\lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|) := \begin{cases} 0 & \text{if } |\alpha| < m + N \\ \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} & \text{if } |\alpha| \geq m + N, \end{cases}$$

we conclude the lemma. □

For the proof of the main Theorem 1.1, we prove the following lemma.

LEMMA 2.2. *For  $a \in \mathbf{D}$  and positive integers  $m$  and  $M$ , we have*

$$\frac{1}{(1-a)^m} = \sum_{k=0}^M \binom{k+m-1}{k} a^k + \sum_{k=1}^m \frac{(k+M+1)_{m-k}}{(m-k)!} \frac{a^{k+M}}{(1-a)^k}.$$

Here  $(M)_k := M(M+1) \cdots (M+k-1)$  denotes the usual Pochhammer symbol for a positive integer  $k$ .

*Proof.* By elementary calculation, we have

$$\frac{1}{(1-a)^m} = \sum_{k=0}^M \binom{k+m-1}{k} a^k + \frac{(M+1)_m}{(m-1)!} a^{M+1} \int_0^1 \frac{(1-t)^M}{(1-at)^{m+M+1}} dt.$$

So we can prove the lemma by showing that

$$(2.1) \quad \int_0^1 \frac{(a-at)^M}{(1-at)^{m+M+1}} dt = \sum_{k=1}^m \frac{(m-1)!a^{k+M-1}}{(M+1)_k(m-k)!(1-a)^k}.$$

If  $a = 0$ , it is trivial. Suppose that  $a \neq 0$ , then

$$\begin{aligned} & \int_0^1 \frac{(a - at)^M}{(1 - at)^{m+M+1}} dt \\ &= \frac{1}{a(1 - a)^m} \int_0^1 \left(1 - \frac{1 - a}{1 - at}\right)^M \left(\frac{1 - a}{1 - at}\right)^{m-1} \frac{a(1 - a)}{(1 - at)^2} dt. \\ &= \frac{1}{a(1 - a)^m} \int_0^a z^M (1 - z)^{m-1} dz, \end{aligned}$$

where we used the change of variables by  $z = 1 - (1 - a)/(1 - at)$ . Define

$$\varphi(i, j) := \int_0^a z^i (1 - z)^j dz$$

for nonnegative integers  $i$  and  $j$ . By integration by parts we obtain

$$\begin{aligned} \varphi(i, j) &= \int_0^a z^i (1 - z)^j dz \\ &= \frac{z^{i+1} (1 - z)^j}{i + 1} \Big|_{z=0}^a + \frac{j}{i + 1} \int_0^a z^{i+1} (1 - z)^{j-1} dz \\ &= \frac{a^{i+1} (1 - a)^j}{i + 1} + \frac{j}{i + 1} \varphi(i + 1, j - 1), \end{aligned}$$

with

$$\varphi(i, 0) = \frac{a^{i+1}}{i + 1}.$$

By solving  $\varphi(i, j)$  defined inductively, we get

$$\varphi(i, j) = \sum_{k=1}^{j+1} \frac{a^{i+k} (1 - a)^{j-k+1} j!}{(i + 1)_k (j - k + 1)!}.$$

Thus we have

$$\int_0^1 \frac{(a - at)^M}{(1 - at)^{m+M+1}} dt = \sum_{k=1}^m \frac{(m - 1)! a^{k+M-1}}{(M + 1)_k (m - k)! (1 - a)^k}.$$

□

### 3. Proof

Now we prove the Theorem 1.1.

**THEOREM 3.1.** *Let  $1 < p < \infty$  and  $N$  be a positive integer. For  $f$  in  $H^p(\mathbf{B}_n)$ ,*

$$f - S_N f = \sum_{m=1}^n c_{m,n,N} T_{m,N}[f],$$

where  $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$ .

*Proof.* Let  $f \in H^p(\mathbf{B}_n)$  with  $1 \leq p < 2$ , then we have

$$f(z) = \int_{\mathbf{S}_n} \frac{f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma_n(\zeta)$$

where  $f^*$  is a radial limit function of  $f$ . So we have

$$f(z) - S_N f(z) = \int_{\mathbf{S}_n} \frac{f^*(\zeta) - S_N f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma_n(\zeta),$$

for a positive integer  $N$ . Here  $S_N$  is  $N$ -th partial sum defined in (1.2). By Lemma 2.2 we have

$$\begin{aligned} f(z) - S_N f(z) &= \sum_{j=0}^N \binom{n-1+j}{j} \int_{\mathbf{S}_n} \langle z, \zeta \rangle^j (f^*(\zeta) - (S_N f)^*(\zeta)) d\sigma_n(\zeta) \\ &+ \frac{(N+1)_n}{(n-1)!} \int_{\mathbf{S}_n} \langle z, \zeta \rangle^{N+1} (f^*(\zeta) - (S_N f)^*(\zeta)) \int_0^1 \frac{(1-t)^N dt}{(1-t\langle z, \zeta \rangle)^{N+n+1}} d\sigma_n(\zeta). \end{aligned}$$

Since  $f^* - (S_N f)^*$  have polynomials with order of greater than  $N$ , the first term equals to zero by orthogonality. Similarly the integration with  $(S_N f)^*$  in the second term is also zero. Combining (2.1) in Lemma 2.2, we get

$$\begin{aligned} f(z) - S_N f(z) &= \frac{(N+1)_n}{(n-1)!} \int_{\mathbf{S}_n} \int_0^1 \frac{(1-t)^N \langle z, \zeta \rangle^{N+1}}{(1-t\langle z, \zeta \rangle)^{N+n+1}} f^*(\zeta) dt d\sigma_n(\zeta) \\ &= (N+1)_n \int_{\mathbf{S}_n} \sum_{m=1}^n \frac{\langle z, \zeta \rangle^{N+m} f^*(\zeta)}{(N+1)_m (n-m)! (1 - \langle z, \zeta \rangle)^m} d\sigma_n(\zeta) \\ &= \sum_{m=1}^n \frac{(m+N+1)_{n-m}}{(n-m)!} \int_{\mathbf{S}_n} \frac{\langle z, \zeta \rangle^{N+k} f^*(\zeta)}{(1 - \langle z, \zeta \rangle)^k} d\sigma_n(\zeta). \end{aligned}$$

Thus we have the relationship between the Szegő type operators and the partial sums of Taylor series as

$$f - S_N f = \sum_{m=1}^n c_{m,n,N} T_{m,N}[f],$$

where  $c_{m,n,N} = \frac{(m+1+N)_{n-m}}{(n-m)!}$ .  $\square$

### References

- [1] P. Duren, *Theory of  $H^p$  spaces*, Academic Press, New York-London, 1970.
- [2] W. Rudin, *Function theory in the unit ball of  $\mathbf{C}^n$* , Springer-Verlag, Berlin, 2008.
- [3] J. Yang, *Norm convergent partial sums of Taylor series*, Bull. Korean Math. Soc. **52** (2015), 1729–1735.
- [4] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer-Verlag, New York, 2005.

Department of Mathematics Education  
Korea National University of Education  
Chungbuk 28173, Republic of Korea  
*E-mail*: jhyang@knue.ac.kr